Appendix

DE-REGRESSION OF EVALUATIONS

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With information available on the breeding values of the sires, the number of the daughters of each sire, the relationship among sires, and the heritability of the trait the phenotypic daughter averages can be reconstructed by de-regression of sire evaluations. This procedure provides an approximation of the average daughter deviation, which is adjusted for all fixed environmental effects.

The appendix shows a method that regenerates the right hand side (RHS) to mixed model equations (MME) with fixed effects absorbed. Furthermore, the computation of the de-regressed proofs will be shown on example records.

1. Material:

The calculations are based on the following records measured for milk yield within one year for 22 progenies of 11 sires in two different herds.

Table 1: First lactation records of 22 cows in kg

Cow	Herd	Sire	Sire of sire	Records in kg
1	1	1		4100
2	2	2		4200
3	1	3		4300
4	1	4	1	4400
5	2	5	1	4500
6	1	6	1	4600
7	2	7	2	4700
8	1	8	2	4800
9	2	9	3	4900
10	1	10	3	5000
11	2	11	3	5100
12	1	4	1	5200
13	2	5	1	5300
14	1	6	1	5400
15	2	7	2	5500
16	1	8	2	5600
17	2	9	3	5700
18	1	10	3	5800
19	2	11	3	5900
20	1	9	3	6000
21	2	10	3	6100
22	1	11	3	6200

2. Least Squares Method

Performing a least squares analysis without any assumptions on variances and co-variances the model is:

$$\mathbf{y=X}\boldsymbol{b}\mathbf{+e}$$

where:

- y: vector with observations
- *b*: vector of unknown parameters
- e: vector with error terms
- X: incidence-matrix indicating for each observation the fixed effects by which it is influenced. It relates parameters to observations in y.

Although the model above shows a least squares equation the matrices are build according to the mixed model equation:

$$\begin{matrix} \boldsymbol{I}. \\ \boldsymbol{II}. \end{matrix} \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} \end{bmatrix} * \begin{bmatrix} \hat{\boldsymbol{b}} \\ \hat{\boldsymbol{u}}_{\boldsymbol{lsq}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'Y} \\ \boldsymbol{Z'Y} \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{\boldsymbol{b}} \\ \hat{\boldsymbol{u}}_{\boldsymbol{lsq}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} \end{bmatrix}^{-} * \begin{bmatrix} \boldsymbol{X'Y} \\ \boldsymbol{Z'Y} \end{bmatrix}$$

As noted above a total of 22 offspring were measured. The resulting matrices and vectors can be written as follows:

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} 4100 \\ 4200 \\ 4300 \\ 4400 \\ 4500 \\ 4600 \\ 4700 \\ 4800 \\ 4900 \\ 5000 \\ 5100 \\ 5200 \\ 5300 \\ 5400 \\ 5500 \\ 5600 \\ 5700 \\ 5800 \\ 5900 \\ 6000 \\ 6100 \\ 6200 \end{bmatrix}$$

$$X'X=\begin{bmatrix}12&0\\0&10\end{bmatrix},\quad Z'Z=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&2&0&0&0&0&0&0&0\\0&0&0&0&2&0&0&0&0&0&0\\0&0&0&0&0&2&0&0&0&0&0\\0&0&0&0&0&0&2&0&0&0&0\\0&0&0&0&0&0&0&2&0&0&0\\0&0&0&0&0&0&0&0&3&0&0\\0&0&0&0&0&0&0&0&0&3&0\\0&0&0&0&0&0&0&0&0&0&3\end{bmatrix},\quad Z'X=\begin{bmatrix}1&0\\0&1\\1&0\\2&0\\0&2\\2&0\\0&2\\2&0\\1&2\\2&1\\1&2\end{bmatrix}$$

$$X'Z=\begin{bmatrix}1&0&1&2&0&2&0&2&1&2&1\\0&1&0&0&2&0&2&0&2&1&1\end{bmatrix},\quad X'Y=\begin{bmatrix}61400\\51900\end{bmatrix},\quad Z'Y=\begin{bmatrix}4100\\4200\\4300\\9600\\9800\\10000\\10200\\10400\\16600\\16900\\17200\end{bmatrix}$$

In models where the X matrix does not have a full rank (some columns equal linear combinations of others), not all linear functions of b can be estimated unbiasedly. If the inverse matrix does not exist, a solution may be written in terms of a generalised inverse. Any generalised inverse can be used to get a solution for $\hat{b}=(X'X)^{-}X'Y$. This can be tested by calculating $k'(X'X)^{-}X'X=k'$. The section can also be called ordinary least squares, because it was assumed that all observations were uncorrelated and had equal variance σ^2_e. Calculated estimates are called least squares estimates. In fact, the estimation is a class mean of the herds ($\hat{\boldsymbol{b}}$) as a deviation from the total mean and the grandsire $(\hat{\boldsymbol{g}}_{\boldsymbol{x}})$ and sire proofs $(\hat{\boldsymbol{s}}_{\boldsymbol{x}})$ show the average daughter deviation, adjusted for the fixed environmental effect of the herds.

$$\begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \\ \hat{s}_6 \\ \hat{s}_7 \\ \hat{s}_8 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 2 & 1 & 2 & 1 \\ 0 & 10 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}^{-1} * \begin{bmatrix} 61400 \\ 51900 \\ 4100 \\ 4200 \\ 4300 \\ 9600 \\ 9800 \\ 10000 \\ 10200 \\ 10400 \\ 16600 \\ 16900 \\ 17200 \end{bmatrix} = \begin{bmatrix} 5053.53 \\ 4820.20 \\ -953.53 \\ -620.20 \\ -753.53 \\ -253.53 \\ 79.79 \\ -53.53 \\ 279.79 \\ 146.46 \\ 635.35 \\ 657.57 \\ 835.35 \end{bmatrix}$$

A further way is to absorb the equation of herd effects into the equation of effects for the grandsire and sire estimates. This procedure is described in the following formula:

$$\textit{\textbf{I}}. \qquad X'X\hat{b} + X'Z\hat{u}_{lsq} = X'Y$$

$$X'X\hat{b} = X'Y - X'Z\hat{u}_{lsq}$$

$$\hat{b} = [X'X]^{-}[X'Y] - [X'X]^{-}[X'Z\hat{u}_{lsq}]$$

$$\hat{b} = [X'X]^{-}[X'Y - X'Z\hat{u}_{lsq}]$$

$$\textit{\textbf{II}}. \qquad Z'X\hat{b} + Z'Z\hat{u}_{lsq} = Z'Y$$

$$Z'X[X'X]^{-}[X'Y - X'Z\hat{u}_{lsq}] + Z'Z\hat{u}_{lsq} = Z'Y$$

$$Z'X[X'X]^{-}X'Y - Z'X[X'X]^{-}X'Z\hat{u}_{lsq} + Z'Z\hat{u}_{lsq} = Z'Y$$

$$[Z'Z - Z'X[X'X]^{-}X'Z]\hat{u}_{lsq} + Z'X[X'X]^{-}X'Y = Z'Y$$

$$\Rightarrow \quad \hat{u}_{lsq} = [Z'Z - Z'X[X'X]^{-}X'Z]^{-}[Z'Y - Z'X[X'X]^{-}X'Y]$$

Solutions for the estimation of class means of the herds ($\hat{b}$) as a deviation from the total mean and the average daughter deviation, adjusted for the fixed environmental effect of the herds for the grandsire $(\hat{g}_x)$ and sire proofs $(\hat{s}_x)$ are similar to calculations from least square estimations above.

$$\begin{bmatrix} \hat{\boldsymbol{g}}_1 \\ \hat{\boldsymbol{g}}_2 \\ \hat{\boldsymbol{g}}_3 \\ \hat{\boldsymbol{s}}_1 \\ \hat{\boldsymbol{s}}_2 \\ \hat{\boldsymbol{s}}_3 \\ \hat{\boldsymbol{s}}_4 \\ \hat{\boldsymbol{s}}_5 \\ \hat{\boldsymbol{s}}_6 \\ \hat{\boldsymbol{s}}_7 \\ \hat{\boldsymbol{s}}_8 \end{bmatrix} = \begin{bmatrix} 11/12 & 0 & -1/12 & -1/6 & 0 & -1/6 & 0 & -1/6 & -1/12 & -1/6 & -1/12 \\ 0 & 9/10 & 0 & 0 & -1/5 & 0 & -1/5 & 0 & -1/5 & -1/10 & -1/5 \\ -1/12 & 0 & 11/12 & -1/6 & 0 & -1/6 & 0 & -1/6 & -1/12 & -1/6 & -1/12 \\ -1/6 & 0 & -1/6 & 1\,2/3 & 0 & -1/3 & 0 & -1/3 & -1/6 & -1/3 & -1/6 \\ 0 & -1/5 & 0 & 0 & 1\,2/5 & 0 & -2/5 & 0 & -2/5 & -1/5 & -2/5 \\ -1/6 & 0 & -1/6 & -1/3 & 0 & 1\,2/3 & 0 & -1/3 & -1/6 & -1/3 & -1/6 \\ 0 & -1/5 & 0 & 0 & -2/5 & 0 & 1\,2/5 & 0 & -2/5 & -1/5 & -2/5 \\ -1/6 & 0 & -1/6 & -1/3 & 0 & -1/3 & 0 & 1\,2/3 & -1/6 & -1/3 & -1/6 \\ -1/12 & -1/5 & -1/12 & -1/6 & -2/5 & -1/6 & -2/5 & -1/6 & 2\,31/60 & -11/30 & -29/60 \\ -1/6 & -1/10 & -1/6 & -1/3 & -1/5 & -1/3 & -1/5 & -1/3 & -11/30 & 2\,17/30 & -11/30 \\ -1/12 & -1/5 & -1/12 & -1/6 & -2/5 & -1/6 & -2/5 & -1/6 & -29/60 & -11/30 & 2\,31/60 \end{bmatrix}^{-1} * \begin{bmatrix} -1016\,2/3 \\ -999 \\ -816\,2/3 \\ -633\,1/3 \\ -580 \\ -233\,1/3 \\ -180 \\ 166\,2/3 \\ 1103\,1/3 \\ 1476\,2/3 \\ 1703\,1/3 \end{bmatrix}$$

$$\boldsymbol{ulsq} = \begin{bmatrix} -953.53 \\ -620.20 \\ -753.53 \\ -253.53 \\ 79.79 \\ -53.53 \\ 279.79 \\ 146.46 \\ 635.35 \\ 657.57 \\ 835.35 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 5053.53 \\ 4820.20 \end{bmatrix}$$

3. BLUP-Method

A mixed model contains fixed and random effects (Mrode, 1996). No variance is assumed on fixed effects, because they contribute equally to all observations. Considering fixed effects, means, like the effect of particular feed, herd or season, are of main interest. In derivating breeding values the main objective is to correct for systematic environmental effects. In contrast, random effects arise from a population of effects. These effects are normally distributed around a mean (μ) with a certain variance (σ^2). Breeding values are random effects.

The model includes fixed effects such as herd and random effects of the sire and grandsire. In Matrix notation, the model for the test evaluation is:

$$\mathbf{y=Xb+Zu+e}$$

where:

y:	vector with observations
b:	vector with fixed effects
u:	vector with random effects
e:	vector with error terms
X:	incidence-matrix indicating for each observation the fixed effects by which it is influenced
Z:	incidence-matrix indicating for each observation the random effects by which it is influenced

Mixed model equations (MME) corresponding to the model are:

$$\begin{matrix} \textit{I.} \\ \textit{II.} \end{matrix} \begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z+A^{-1}\lambda \end{bmatrix} * \begin{bmatrix} \hat{b} \\ \hat{u}_{BLUP} \end{bmatrix} = \begin{bmatrix} X'Y \\ Z'Y \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{b} \\ \hat{u}_{BLUP} \end{bmatrix} = \begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z+A^{-1}\lambda \end{bmatrix}^{-1} * \begin{bmatrix} X'Y \\ Z'Y \end{bmatrix}$$

where A^{-1} is the inverse of the relationship matrix among sons and grandsires.

$$A^{-1} = \begin{bmatrix} 2 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\frac{2}{3} & 0 & 0 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 0 & 0 & 1\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 & 1\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 & 0 & 1\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 1\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 1\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 1\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 1\frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\frac{1}{3} \end{bmatrix}$$

From the ratio between the residual (σ^2_e) and sire variance (σ^2_s) λ can be calculated. Assuming a paternal half-sib family structure and a sire model with a heritability of 0.25, $\lambda = (4\text{-}h^2)/h^2$ and becomes 15. The variance-covariance structure of the residual error is R with value $(\boldsymbol{Z}'\boldsymbol{Z} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Z})^{-1}(\sigma^2_e)$.

Solving this system of mixed model equations, best linear estimates (accurate and unbiased) for fixed effects (BLUE) and for breeding values of the grandsires and sires (BLUP) are obtained simultaneously.

$$\begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \\ \hat{s}_6 \\ \hat{s}_7 \\ \hat{s}_8 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 2 & 1 & 2 & 1 \\ 0 & 10 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 1 & 2 \\ 1 & 0 & 31 & 0 & 0 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 26 & 0 & 0 & 0 & 0 & -10 & -10 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 31 & 0 & 0 & 0 & 0 & 0 & -10 & -10 & -10 \\ 2 & 0 & -10 & 0 & 0 & 22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -10 & 0 & 0 & 0 & 22 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -10 & 0 & 0 & 0 & 0 & 22 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -10 & 0 & 0 & 0 & 0 & 22 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 22 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 23 & 0 & 0 \\ 2 & 1 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 23 & 0 \\ 1 & 2 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 23 \end{bmatrix}^{-1} * \begin{bmatrix} 61400 \\ 51900 \\ 4100 \\ 4200 \\ 4300 \\ 9600 \\ 9800 \\ 10000 \\ 10200 \\ 10400 \\ 16600 \\ 16900 \\ 17200 \end{bmatrix} = \begin{bmatrix} 5113.03 \\ 5170.64 \\ -94.81 \\ -56.52 \\ 61.12 \\ -71.55 \\ -67.70 \\ -53.37 \\ -32.11 \\ -17.78 \\ 76.38 \\ 91.93 \\ 102.47 \end{bmatrix}$$

Absorbing the equation for fixed effects $(\hat{\boldsymbol{b}})$ into the equation for grandsire and sire effects $(\hat{\boldsymbol{u}}_{\boldsymbol{BLUP}})$ leads to the following equations and the same results as shown from the MME.

I.

$$\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{b}} + \boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} = \boldsymbol{X}'\boldsymbol{Y}$$
$$\boldsymbol{X}'\boldsymbol{X}\hat{\boldsymbol{b}} = \boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP}$$
$$\hat{\boldsymbol{b}} = [\boldsymbol{X}'\boldsymbol{X}]^{-}[\boldsymbol{X}'\boldsymbol{Y}] - [\boldsymbol{X}'\boldsymbol{X}]^{-}[\boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP}]$$
$$\hat{\boldsymbol{b}} = [\boldsymbol{X}'\boldsymbol{X}]^{-}[\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP}]$$

II.

$$\boldsymbol{Z}'\boldsymbol{X}\hat{\boldsymbol{b}} + \boldsymbol{Z}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} + \boldsymbol{A}^{-1}\lambda\hat{\boldsymbol{u}}_{BLUP} = \boldsymbol{Z}'\boldsymbol{Y}$$
$$\boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}[\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP}] + \boldsymbol{Z}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} + \boldsymbol{A}^{-1}\lambda\hat{\boldsymbol{u}}_{BLUP} = \boldsymbol{Z}'\boldsymbol{Y}$$
$$\boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} + \boldsymbol{Z}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} + \boldsymbol{A}^{-1}\lambda\hat{\boldsymbol{u}}_{BLUP} = \boldsymbol{Z}'\boldsymbol{Y}$$
$$\boldsymbol{Z}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{BLUP} + \boldsymbol{A}^{-1}\lambda\hat{\boldsymbol{u}}_{BLUP} = \boldsymbol{Z}'\boldsymbol{Y} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Y}$$
$$[\boldsymbol{Z}'\boldsymbol{Z} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Z} + \boldsymbol{A}^{-1}\lambda]\hat{\boldsymbol{u}}_{BLUP} = \boldsymbol{Z}'\boldsymbol{Y} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Y}$$

$$\hat{\boldsymbol{u}}_{BLUP} = [\boldsymbol{Z}'\boldsymbol{Z} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Z} + \boldsymbol{A}^{-1}\lambda]^{-1}[\boldsymbol{Z}'\boldsymbol{Y} - \boldsymbol{Z}'\boldsymbol{X}[\boldsymbol{X}'\boldsymbol{X}]^{-}\boldsymbol{X}'\boldsymbol{Y}]$$

$$
\begin{bmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \\ \hat{s}_6 \\ \hat{s}_7 \\ \hat{s}_8 \end{bmatrix}
=
\begin{bmatrix}
30\tfrac{11}{12} & 0 & -\tfrac{1}{12} & -10\tfrac{1}{6} & -10 & -10\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{1}{12} \\
0 & 25\tfrac{9}{10} & 0 & 0 & -\tfrac{1}{5} & 0 & -10\tfrac{1}{5} & -10 & -\tfrac{1}{5} & -\tfrac{1}{10} & -\tfrac{1}{5} \\
-\tfrac{1}{12} & 0 & 30\tfrac{11}{12} & -\tfrac{1}{6} & 0 & -\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -10\tfrac{1}{12} & -10\tfrac{1}{6} & -10\tfrac{1}{12} \\
-10\tfrac{1}{6} & 0 & -\tfrac{1}{6} & 21\tfrac{2}{3} & 0 & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} \\
-10 & -\tfrac{1}{5} & 0 & 0 & 21\tfrac{3}{5} & 0 & -\tfrac{2}{5} & 0 & -\tfrac{2}{5} & -\tfrac{1}{5} & -\tfrac{2}{5} \\
-10\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -\tfrac{1}{3} & 0 & 21\tfrac{2}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} \\
0 & -10\tfrac{1}{5} & 0 & 0 & -\tfrac{2}{5} & 0 & 21\tfrac{3}{5} & 0 & -\tfrac{2}{5} & -\tfrac{1}{5} & -\tfrac{2}{5} \\
-\tfrac{1}{6} & -10 & -\tfrac{1}{6} & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} & 0 & 21\tfrac{2}{3} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} \\
-\tfrac{1}{12} & -\tfrac{1}{5} & -10\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & 22\tfrac{31}{60} & -\tfrac{11}{30} & -\tfrac{29}{60} \\
-\tfrac{1}{6} & -\tfrac{1}{10} & -10\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{5} & -\tfrac{1}{3} & -\tfrac{1}{5} & -\tfrac{1}{3} & -\tfrac{11}{30} & 22\tfrac{17}{30} & -\tfrac{11}{30} \\
-\tfrac{1}{12} & -\tfrac{1}{5} & -10\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & -\tfrac{29}{60} & -\tfrac{11}{30} & 22\tfrac{31}{60}
\end{bmatrix}^{-1}
*
\begin{bmatrix} -1016\tfrac{2}{3} \\ -999 \\ -816\tfrac{2}{3} \\ -633\tfrac{1}{3} \\ -580 \\ -233\tfrac{1}{3} \\ -180 \\ 166\tfrac{2}{3} \\ 1103\tfrac{1}{3} \\ 1476\tfrac{2}{3} \\ 1703\tfrac{1}{3} \end{bmatrix}
$$

Results for $(\hat{u}_{BLUP})$ and the respective $(\hat{b})$:

$$
\hat{u}_{BLUP} = \begin{bmatrix} -94.81 \\ -56.52 \\ 61.12 \\ -71.55 \\ -67.70 \\ -53.37 \\ -32.11 \\ -17.78 \\ 76.38 \\ 91.93 \\ 102.47 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 5113.03 \\ 5170.64 \end{bmatrix}
$$

Diagonal elements of $(Z'Z - Z'X[X'X]^{-}X'Z)$ represent the number of effective daughters corresponding to each grandsire and sire.

$$
Z'Z - Z'X(X'X)^{-}X'Z =
\begin{bmatrix}
\tfrac{11}{12} & 0 & -\tfrac{1}{12} & -\tfrac{1}{6} & 0 & -\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{1}{12} \\
0 & \tfrac{9}{10} & 0 & 0 & -\tfrac{1}{5} & 0 & -\tfrac{1}{5} & 0 & -\tfrac{1}{5} & -\tfrac{1}{10} & -\tfrac{1}{5} \\
-\tfrac{1}{12} & 0 & \tfrac{11}{12} & -\tfrac{1}{6} & 0 & -\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{1}{12} \\
-\tfrac{1}{6} & 0 & -\tfrac{1}{6} & 1\tfrac{2}{3} & 0 & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} \\
0 & -\tfrac{1}{5} & 0 & 0 & 1\tfrac{3}{5} & 0 & -\tfrac{2}{5} & 0 & -\tfrac{2}{5} & -\tfrac{1}{5} & -\tfrac{2}{5} \\
-\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -\tfrac{1}{3} & 0 & 1\tfrac{2}{3} & 0 & -\tfrac{1}{3} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} \\
0 & -\tfrac{1}{5} & 0 & 0 & -\tfrac{2}{5} & 0 & 1\tfrac{3}{5} & 0 & -\tfrac{2}{5} & -\tfrac{1}{5} & -\tfrac{2}{5} \\
-\tfrac{1}{6} & 0 & -\tfrac{1}{6} & -\tfrac{1}{3} & 0 & -\tfrac{1}{3} & 0 & 1\tfrac{2}{3} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{6} \\
-\tfrac{1}{12} & -\tfrac{1}{5} & -\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & 2\tfrac{31}{60} & -\tfrac{11}{30} & -\tfrac{29}{60} \\
-\tfrac{1}{6} & -\tfrac{1}{10} & -\tfrac{1}{6} & -\tfrac{1}{3} & -\tfrac{1}{5} & -\tfrac{1}{3} & -\tfrac{1}{5} & -\tfrac{1}{3} & -\tfrac{11}{30} & 2\tfrac{17}{30} & -\tfrac{11}{30} \\
-\tfrac{1}{12} & -\tfrac{1}{5} & -\tfrac{1}{12} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & -\tfrac{2}{5} & -\tfrac{1}{6} & -\tfrac{29}{60} & -\tfrac{11}{30} & 2\tfrac{31}{60}
\end{bmatrix}
$$

Offdiagonal elements of $(Z'Z - Z'X[X'X]^{-}X'Z)$ are small compared with the diagonal elements.

$$\mathbf{Z'X}\hat{\mathbf{b}}+\mathbf{Z'Z}\hat{\mathbf{u}}_{BLUP}+\mathbf{A}^{-1}\lambda\,\hat{\mathbf{u}}_{BLUP}=\mathbf{Z'Y}$$

$$\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Y}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}\hat{\mathbf{u}}_{BLUP}+\mathbf{Z'Z}\hat{\mathbf{u}}_{BLUP}+\mathbf{A}^{-1}\lambda\,\hat{\mathbf{u}}_{BLUP}=\mathbf{Z'Y}$$

$$\mathbf{Z'Z}\hat{\mathbf{u}}_{BLUP}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}\,\hat{\mathbf{u}}_{BLUP}=\mathbf{Z'Y}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Y}-\mathbf{A}^{-1}\lambda\hat{\mathbf{u}}_{BLUP}$$

$$[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]\,\hat{\mathbf{u}}_{BLUP}=\mathbf{Z'Y}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Y}-\mathbf{A}^{-1}\lambda\hat{\mathbf{u}}_{BLUP}$$

$$[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]\hat{\mathbf{u}}_{BLUP}+\mathbf{A}^{-1}\lambda\,\hat{\mathbf{u}}_{BLUP}=\mathbf{Z'Y}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Y}$$

$$\hat{\mathbf{u}}_{BLUP}+[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]^{-1}\mathbf{A}^{-1}\lambda\,\hat{\mathbf{u}}_{BLUP}=[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]^{-1}[\mathbf{Z'Y}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Y}]$$

$$[\mathbf{I}+[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]^{-1}\mathbf{A}^{-1}\lambda]\,\hat{\mathbf{u}}_{BLUP}=\underbrace{[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]^{-1}[\mathbf{Z'Y}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Y}]}$$

$$[\mathbf{I}+[\mathbf{Z'Z}-\mathbf{Z'X}(\mathbf{X'X})^{-}\mathbf{X'Z}]^{-1}\mathbf{A}^{-1}\lambda]\,\hat{\mathbf{u}}_{BLUP}=\hat{\mathbf{u}}_{LSQ}$$

Results from this calculation show the reconstruction of the least squares estimates.

$$\left(\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&1\end{bmatrix}+\right.$$

$$\left.\left(\begin{bmatrix}\tfrac{11}{12}&0&-\tfrac{1}{12}&-\tfrac{1}{6}&0&-\tfrac{1}{6}&0&-\tfrac{1}{6}&-\tfrac{1}{12}&-\tfrac{1}{6}&-\tfrac{1}{12}\\0&\tfrac{9}{10}&0&0&-\tfrac{1}{5}&0&-\tfrac{1}{5}&0&-\tfrac{1}{5}&-\tfrac{1}{10}&-\tfrac{1}{5}\\-\tfrac{1}{12}&0&\tfrac{11}{12}&-\tfrac{1}{6}&0&-\tfrac{1}{6}&0&-\tfrac{1}{6}&-\tfrac{1}{12}&-\tfrac{1}{6}&-\tfrac{1}{12}\\-\tfrac{1}{6}&0&-\tfrac{1}{6}&1\tfrac{2}{3}&0&-\tfrac{1}{3}&0&-\tfrac{1}{3}&-\tfrac{1}{6}&-\tfrac{1}{3}&-\tfrac{1}{6}\\0&-\tfrac{1}{5}&0&0&1\tfrac{2}{5}&0&-\tfrac{2}{5}&0&-\tfrac{2}{5}&-\tfrac{1}{5}&-\tfrac{2}{5}\\-\tfrac{1}{6}&0&-\tfrac{1}{6}&-\tfrac{1}{3}&0&1\tfrac{2}{3}&0&-\tfrac{1}{3}&-\tfrac{1}{6}&-\tfrac{1}{3}&-\tfrac{1}{6}\\0&-\tfrac{1}{5}&0&0&-\tfrac{2}{5}&0&1\tfrac{2}{5}&0&-\tfrac{2}{5}&-\tfrac{1}{5}&-\tfrac{2}{5}\\-\tfrac{1}{6}&0&-\tfrac{1}{6}&-\tfrac{1}{3}&0&-\tfrac{1}{3}&0&1\tfrac{2}{3}&-\tfrac{1}{6}&-\tfrac{1}{3}&-\tfrac{1}{6}\\-\tfrac{1}{12}&-\tfrac{1}{5}&-\tfrac{1}{12}&-\tfrac{1}{6}&-\tfrac{2}{5}&-\tfrac{1}{6}&-\tfrac{2}{5}&-\tfrac{1}{6}&2\tfrac{31}{60}&-\tfrac{11}{30}&-\tfrac{29}{60}\\-\tfrac{1}{6}&-\tfrac{1}{10}&-\tfrac{1}{6}&-\tfrac{1}{3}&-\tfrac{1}{5}&-\tfrac{1}{3}&-\tfrac{1}{5}&-\tfrac{1}{3}&-\tfrac{11}{30}&2\tfrac{17}{30}&-\tfrac{11}{30}\\-\tfrac{1}{12}&-\tfrac{1}{5}&-\tfrac{1}{12}&-\tfrac{1}{6}&-\tfrac{2}{5}&-\tfrac{1}{6}&-\tfrac{2}{5}&-\tfrac{1}{6}&-\tfrac{29}{60}&-\tfrac{11}{30}&2\tfrac{31}{60}\end{bmatrix}^{-1}\begin{bmatrix}2&0&0&-\tfrac{2}{3}&-\tfrac{2}{3}&-\tfrac{2}{3}&0&0&0&0&0\\0&1\tfrac{2}{3}&0&0&0&0&-\tfrac{2}{3}&-\tfrac{2}{3}&0&0&0\\0&0&2&0&0&0&0&0&-\tfrac{2}{3}&-\tfrac{2}{3}&-\tfrac{2}{3}\\-\tfrac{2}{3}&0&0&1\tfrac{1}{3}&0&0&0&0&0&0&0\\-\tfrac{2}{3}&0&0&0&1\tfrac{1}{3}&0&0&0&0&0&0\\-\tfrac{2}{3}&0&0&0&0&1\tfrac{1}{3}&0&0&0&0&0\\0&-\tfrac{2}{3}&0&0&0&0&1\tfrac{1}{3}&0&0&0&0\\0&-\tfrac{2}{3}&0&0&0&0&0&1\tfrac{1}{3}&0&0&0\\0&0&-\tfrac{2}{3}&0&0&0&0&0&1\tfrac{1}{3}&0&0\\0&0&-\tfrac{2}{3}&0&0&0&0&0&0&1\tfrac{1}{3}&0\\0&0&-\tfrac{2}{3}&0&0&0&0&0&0&0&1\tfrac{1}{3}\end{bmatrix}^{-1}*15\right)\right)$$

$$*\begin{bmatrix}-94.81\\-56.52\\61.12\\-71.55\\-67.70\\-53.37\\-32.11\\-17.78\\76.38\\91.93\\102.47\end{bmatrix}=\begin{bmatrix}-953.53\\-620.20\\-753.53\\-253.53\\79.79\\-53.53\\279.79\\146.46\\635.35\\657.57\\835.35\end{bmatrix}\Rightarrow\ \hat{\mathbf{u}}_{lsq}$$

Using the method of Lien et al. (1995) to regenerate the right hand side (RHS) to MME with fixed effects absorbed will result in the following equations. Because offdiagonal elements are small compared with the diagonal elements they can be assumed to be null. After premultiplication by $(\mathbf{Z'Z}-\mathbf{Z'X}[\mathbf{X'X}]^{-}\mathbf{X'Z})^{-1}$ the regenerated RHS to MME with the fixed effects absorbed results in a measurement of the daughter yield deviations, which account for fixed effects in the model and will be expressed in phenotypic units.

$$\left(\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} + \left(\begin{bmatrix} \tfrac{11}{12}&0&0&0&0&0&0&0&0&0&0\\ 0&\tfrac{9}{10}&0&0&0&0&0&0&0&0&0\\ 0&0&\tfrac{11}{12}&0&0&0&0&0&0&0&0\\ 0&0&0&1\tfrac{2}{3}&0&0&0&0&0&0&0\\ 0&0&0&0&1\tfrac{2}{5}&0&0&0&0&0&0\\ 0&0&0&0&0&1\tfrac{2}{3}&0&0&0&0&0\\ 0&0&0&0&0&0&1\tfrac{2}{5}&0&0&0&0\\ 0&0&0&0&0&0&0&1\tfrac{2}{3}&0&0&0\\ 0&0&0&0&0&0&0&0&2\tfrac{31}{60}&0&0\\ 0&0&0&0&0&0&0&0&0&2\tfrac{17}{30}&0\\ 0&0&0&0&0&0&0&0&0&0&2\tfrac{31}{60} \end{bmatrix}^{-1} \begin{bmatrix} 2&0&0&-\tfrac{2}{3}&-\tfrac{2}{3}&-\tfrac{2}{3}&0&0&0&0&0\\ 0&1\tfrac{2}{3}&0&0&0&0&-\tfrac{2}{3}&-\tfrac{2}{3}&0&0&0\\ 0&0&2&0&0&0&0&0&-\tfrac{2}{3}&-\tfrac{2}{3}&-\tfrac{2}{3}\\ -\tfrac{2}{3}&0&0&1\tfrac{1}{3}&0&0&0&0&0&0&0\\ -\tfrac{2}{3}&0&0&0&1\tfrac{1}{3}&0&0&0&0&0&0\\ -\tfrac{2}{3}&0&0&0&0&1\tfrac{1}{3}&0&0&0&0&0\\ 0&-\tfrac{2}{3}&0&0&0&0&1\tfrac{1}{3}&0&0&0&0\\ 0&-\tfrac{2}{3}&0&0&0&0&0&1\tfrac{1}{3}&0&0&0\\ 0&0&-\tfrac{2}{3}&0&0&0&0&0&1\tfrac{1}{3}&0&0\\ 0&0&-\tfrac{2}{3}&0&0&0&0&0&0&1\tfrac{1}{3}&0\\ 0&0&-\tfrac{2}{3}&0&0&0&0&0&0&0&1\tfrac{1}{3} \end{bmatrix}^{-1} * 15\right)\right)$$

$$* \begin{bmatrix} -94.81\\ -56.52\\ 61.12\\ -71.55\\ -67.70\\ -53.37\\ -32.11\\ -17.78\\ 76.38\\ 91.93\\ 102.47 \end{bmatrix} = \begin{bmatrix} -1096.50\\ -1072.22\\ -892.50\\ -361.32\\ -357.62\\ -124.96\\ -87.16\\ 107.91\\ 440.56\\ 570.17\\ 673.96 \end{bmatrix}$$

As a further extension the calculation of the Least Squares estimates was derived without consideration of the “fixed” effects.

$$\begin{matrix} \boldsymbol{I}. \\ \boldsymbol{II}. \end{matrix} \quad \begin{bmatrix} \boldsymbol{X'X} & \boldsymbol{X'Z} \\ \boldsymbol{Z'X} & \boldsymbol{Z'Z} \end{bmatrix} * \begin{bmatrix} \hat{\boldsymbol{b}} \\ \hat{\boldsymbol{u}}_{\boldsymbol{lsq}\,(\boldsymbol{red})} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X'Y} \\ \boldsymbol{Z'Y} \end{bmatrix}$$

$$\boldsymbol{II}.(\boldsymbol{red}) \qquad \boldsymbol{Z'Z}\hat{\boldsymbol{u}}_{\boldsymbol{lsq}\,(\boldsymbol{red})} = \boldsymbol{Z'Y}$$

$$\hat{\boldsymbol{u}}_{\boldsymbol{lsq}\,(\boldsymbol{red})} = [\boldsymbol{Z'Z}]^{-1}\boldsymbol{Z'Y}$$

Results of this calculation show the average performance of the daughters of the respective grandsires or sires.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tfrac{1}{3} \end{bmatrix} * \begin{bmatrix} 4100 \\ 4200 \\ 4300 \\ 9600 \\ 9800 \\ 10000 \\ 10200 \\ 10400 \\ 16600 \\ 16900 \\ 17200 \end{bmatrix} \mathrel{\widehat{=}} \begin{bmatrix} 4100 \\ 4200 \\ 4300 \\ 4800 \\ 4900 \\ 5000 \\ 5100 \\ 5200 \\ 5533\tfrac{1}{3} \\ 5633\tfrac{1}{3} \\ 5733\tfrac{1}{3} \end{bmatrix} = \hat{\boldsymbol{u}}_{\boldsymbol{lsq}\,(\boldsymbol{red})}$$

Estimation of breeding values without preadjustment for fixed or environmental effects will result in biased breeding values, because breeding values in this example are also influenced by herd effects. Although no correction for fixed effects was considered, the following equations make clear the relationship between LSE and BLUP.

$$\begin{matrix} \boldsymbol{I}. \\ \boldsymbol{II}. \end{matrix} \quad \begin{bmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} + \boldsymbol{A}^{-1}\lambda \end{bmatrix} * \begin{bmatrix} \hat{\boldsymbol{b}} \\ \hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{X}'\boldsymbol{Y} \end{bmatrix}$$

$$\boldsymbol{II}.(\boldsymbol{red}) \qquad \begin{aligned} \boldsymbol{Z}'\boldsymbol{Z}\hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} + \boldsymbol{A}^{-1}\lambda\,\hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} &= \boldsymbol{Z}'\boldsymbol{Y} \\ \hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} + [\boldsymbol{Z}'\boldsymbol{Z}]^{-1}\boldsymbol{A}^{-1}\lambda\,\hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} &= [\boldsymbol{Z}'\boldsymbol{Z}]^{-1}\boldsymbol{Z}'\boldsymbol{Y} \\ [\boldsymbol{I} + [\boldsymbol{Z}'\boldsymbol{Z}]^{-1}\boldsymbol{A}^{-1}\lambda]\,\hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} &= [\boldsymbol{Z}'\boldsymbol{Z}]^{-1}\boldsymbol{Z}'\boldsymbol{Y} \\ [\boldsymbol{I} + [\boldsymbol{Z}'\boldsymbol{Z}]^{-1}\boldsymbol{A}^{-1}\lambda]\,\hat{\boldsymbol{u}}_{\boldsymbol{BLUP}\,(\boldsymbol{red})} &= \hat{\boldsymbol{u}}_{\boldsymbol{lsq}\,(\boldsymbol{red})} \end{aligned}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 30 & 0 & 0 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 & -10 & -10 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 & 0 & 0 & 0 & -10 & -10 & -10 \\ -5 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & -3\tfrac{1}{3} & 0 & 0 & 0 & 0 & 0 & 6\tfrac{2}{3} & 0 & 0 \\ 0 & 0 & -3\tfrac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 6\tfrac{2}{3} & \\ 0 & 0 & -3\tfrac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\tfrac{2}{3} \end{bmatrix} \right) * \begin{bmatrix} 1005.75 \\ 802.15 \\ 1467.07 \\ 893.52 \\ 902.61 \\ 911.70 \\ 828.25 \\ 837.34 \\ 1359.59 \\ 1372.63 \\ 1385.68 \end{bmatrix} = \begin{bmatrix} 4100 \\ 4200 \\ 4300 \\ 4800 \\ 4900 \\ 5000 \\ 5100 \\ 5200 \\ 5533\tfrac{1}{3} \\ 5633\tfrac{1}{3} \\ 5733\tfrac{1}{3} \end{bmatrix}$$

Literature

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